$x' = x^2$, $y = y$ to the parabola $y = x'^2 + qx' + s$, thus obtaining the auxiliary quartic curve. Note that the vertex of the parabola is $(-q/2, s - q^2/4)$ and that the parabola and quartic have in common the y-intercept s.

The different forms of the quartic curve depend on the position of the parabola relative to the y-axis. The three cases follow.

$q < 0$, the vertex of the parabola is to right of the y-axis; the quartic has three real and distinct turning points, $(0, s)$, $\left(\pm\sqrt{-\frac{q}{2}}, s - \frac{q^2}{4}\right)$.

$q = 0$, the vertex of the parabola is on the y-axis; the quartic has a triple turning point $(0, s)$.

$q > 0$, the vertex of the parabola is to left of the y-axis; the quartic has only one real turning point $(0, s)$.

The arguments for the various cases of the theorem with few exceptions are identical with those sketched above.[1]

TWO NEW CONSTRUCTIONS OF THE STROPHOID.

By R. M. MATHEWS, Wesleyan University.

(Read before the American Mathematical Society December 28, 1920.)

1. The classic construction for the strophoid uses a pencil of circles each of which has its center on a "medial" line g and passes through a fixed point, the node O, on g (Fig. 1). Let each circle be cut by that one of its diameters which passes through a fixed point, the singular focus F. The curve is the locus of these intersections.[2] The object of this note is to make this construction more general for the same curve: first, by using any line through the node as locus for the centers of the circles; and second, by using a pencil of circles through any two conjugate points of the curve. In preparation for this we describe certain well known features of the curve.[3]

[1] Instead of adding the ordinates of the line $y = -rx$ and the curve $y = f(x)$, the author might have started with the curve $y = x^4 + qx^2 + s$ and regarded the roots of the given quartic as the abscissas of the intersections of this curve and the line $y = -rx$. The form of this curve depends only on q; its position, or the position of the origin with respect to it, depends on s, while the character of the roots of the equation, when q and s are given, depends on r. Thus the classification, based first on q, and then on s, would finally be based on r.

The range of values of r for any type of equation, when q and s are given, depends on those values which correspond to the real tangents from the origin. These values of r are the roots of the equation $\Delta = 0$, and for any particular type of equation Δ will have a particular sign or be zero. Conversely, the sign or vanishing of Δ, with the given values of q and s, will usually determine the type of the equation. These considerations would enable us to dispense with the author's theorem on discriminants. Results obtained as depending on r could be interpreted at once as depending on Δ, and so when the classification has been obtained, the various classes could be grouped and arranged with respect to Δ, q and s if such an arrangement is more convenient for use.—EDITOR.

[2] Gino Loria, *Spezielle algebraische und transcendente ebene Kurven*, volume 1, Leipzig, 1910, p. 60. The strophoid of our text-books is the *right* strophoid, the form this curve takes when the node is at the foot of the perpendicular from the focus to the median.

[3] Loria, *loc. cit.*, chapter 8.

2. The strophoid is *the* orthotomic circular cubic. With the tangents at the node as axes, its equation may be written

$$(x^2 + y^2)(y + cx) - axy = 0;$$

or in parametric form

$$x = \frac{am}{(1 + m^2)(m + c)}, \qquad y = mx. \tag{1}$$

The real asymptote is parallel to the medial line g: $y + cx = 0$; while the two imaginary asymptotes meet at the singular focus F which is on the line $y - cx = 0$. the *axis* of the curve.

The nodal tangents bisect the angles formed by the axis and the median.

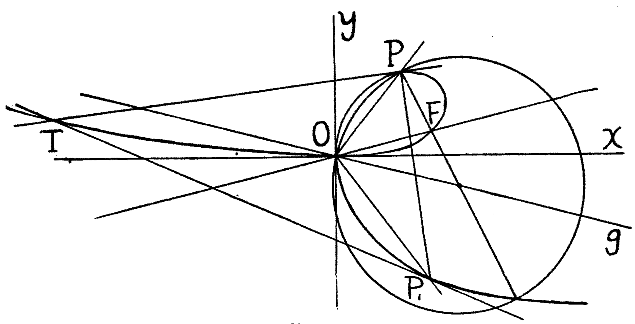


FIG. 1.

Two points, P and P_1, whose parameters are m and $-m$, are "conjugate" points; that is, the tangents at these points meet the curve at the same "tangential point" T. The medial line bisects the join of every pair of conjugate points. Evidently, the nodal tangents bisect the angles determined at the node by each pair of conjugate points.

On substituting the parametric values of x and y in the equation $ux + vy + 1 = 0$ we find that the necessary and sufficient condition that three points m_1, m_2, m_3 of the strophoid be collinear is $m_1m_2m_3 = -c$.

3. Let us consider the pencil of circles of parameter h which are specified by the equation

$$x^2 + y^2 - 2hx - 2lhy = 0. \tag{2}$$

Each circle passes through the node O (Fig. 2), and has its center (h, lh) on the line $y = lx$, which we may suppose to be the line OP. When this equation is solved simultaneously with that of the strophoid, we obtain, besides the node counted twice and the circular points at infinity, the points whose parameters are roots of the equation

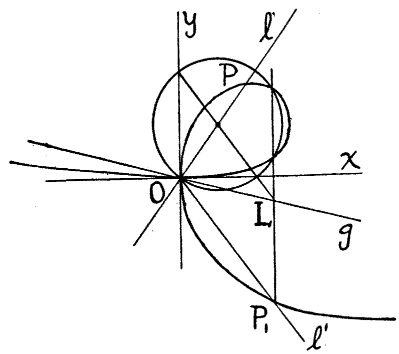


FIG. 2.

$$m^2 + \frac{1}{2lh}(2h + 2lhc - a)m + \frac{c}{l} = 0.$$

Hence $m_1m_2 = c/l$, a relation which implies $m_1m_2(-l) = -c$, and shows that

the points of intersection are collinear with the point P_1 which is conjugate to P. Thus the curve appears as the intersection of a pencil of lines with a projective pencil of circles. To correlate the two forms, we find that the join of m_1 and m_2 cuts the medial line $y + cx = 0$ in the point L

$$L: \qquad x = \frac{2lh}{l - c}, \qquad y = \frac{-2clh}{l - c}.$$

This result justifies the following construction of the curve, given the node O and two conjugate points P, P_1. Construct the tangents at the node and the medial line. Take the line OP for the line $y = lx$. The circle (2) through O with its center on this line cuts the axes again in $(2h, 0)$ and $(0, 2lh)$. The join of these points cuts the medial line in the point L, and the line P_1L cuts the circle in the desired points of the strophoid.

If we take l as the medial line, then P_1 is the singular focus and we have the classical construction.

4. Another construction can be obtained by considering the pencil of circles through two distinct conjugate points P and P_1 of parameters m and $-m$. A circle

$$x^2 + y^2 - 2hx - 2ky + d = 0 \tag{3}$$

cuts the strophoid in the circular points at infinity and at the points whose parameters are roots of the equation

$$dm^4 + 2(dc - ak)m^3 + (d + dc^2 + a^2 - 2ah - 2ack)m^2 + 2c(d - ah)m + dc^2 = 0.$$

Thus the necessary and sufficient condition that four points be concyclic is

$$m_1m_2m_3m_4 = c^2.$$

Taking m and $-m$ for m_1 and m_2 we have the pencil of circles on PP_1 such that

$$m_3m_4(m^2/c) = -c.$$

Therefore, every circle through the conjugate points of parameters m, $-m$ cuts the strophoid in two points Q, Q_1 which are collinear with a certain point K of parameter m^2/c. (Fig. 3.)

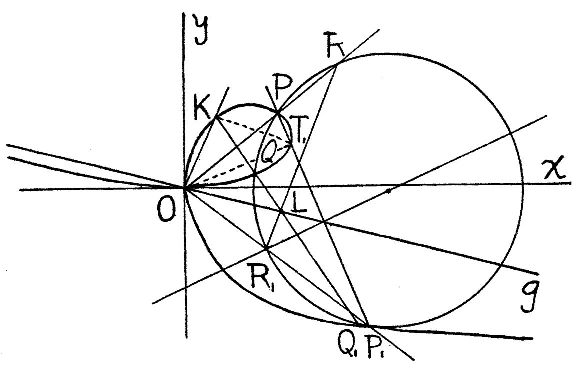


FIG. 3.

From the parametric equations it is easy to show that the slope of PP_1 is $-m^2/c$. The perpendicular from the origin upon this line meets it at the point c/m^2 of the strophoid, say T_1 (for it is the conjugate to the point T, the common tangential of P and P_1). The point at infinity of the strophoid has for parameter $-c$ and is collinear

with K and T_1, since $(m^2/c)(c/m^2)(-c) = -c$. Therefore to construct K we draw OT_1 perpendicular to PP_1, OK the reflection of OT_1 around the bisector of the angle between the nodal tangents, and T_1K parallel to the medial line g.

It remains to put the pencil of circles on PP_1 into graphical correspondence with the pencil of lines at K. Each circle through P and P_1 cuts the nodal radii OP and OP_1 again in two points R and R_1. To determine R, substitute $y = mx$ in (3). The result is a quadratic in x the product of whose roots is $d/(1 + m^2)$. But one of these is the x in (1); therefore the other root is $d(m + c)/am$. In this way we find the coördinates of R and R_1 to be:

$$\left[\frac{d}{am}(m + c), \quad \frac{d}{a}(m + c)\right] \quad \text{and} \quad \left[\frac{d}{am}(m - c), \quad \frac{-d}{a}(m - c)\right].$$

The equation of the line RR_1 is then

$$am^2x - acy - d(m^2 - c^2) = 0,$$

and it requires only some algebraic drudgery to show that this line meets KQQ_1 on the medial line g.

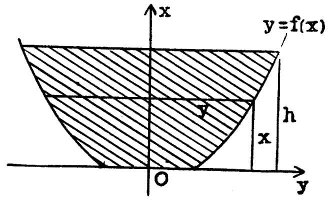
Accordingly, the strophoid may be constructed as follows, given the node and two conjugate points P and P_1. Construct the nodal tangents, the medial line, the line OK and the point K where it will cut the curve. Each circle of the pencil through PP_1 cuts the nodal radii OP, OP_1 in two points R, R_1; the line RR_1 cuts the medial line in a point L, and the line LK cuts the circle in its remaining real intersections with the strophoid.

5. While these constructions are not superior to the classical one in case of actual use on the drawing board, they are of importance as bases for the study of new properties of the curve.

AN APPLICATION OF ABEL'S INTEGRAL EQUATION.

By W. C. BRENKE, University of Nebraska.

Let the shaded area in the figure represent the cross section of a weir notch, the cross section being symmetrical with respect to the x-axis. The quantity of flow through the notch per unit time will be given by



$$Q = C\int_0^h \sqrt{h - x}\, f(x)dx,$$

where the form of the notch is determined by $y = f(x)$; $x \geqq 0$.

Consider the problem of determining $f(x)$ so that the quantity of flow per unit of time shall be proportional to a given power of the depth of stream; *i.e.*, $Q = k'h^m$, $m > 0$. Hence we must find $f(x)$ from an integral equation of the form

$$\int_0^h \sqrt{h - x}\, f(x)dx = kh^m. \qquad (1)$$